The stream function of potential theory for a dual-pipe subirrigation-drainage system

Don Kirkham
Iowa State University

Robert Horton Jr.
Iowa State University, rhorton@iastate.edu

The stream function of potential theory for a dual-pipe subirrigation-drainage system

Abstract

An exact mathematical solution to Laplace's equation is presented for appropriate boundary conditions associated with the problem of dual-pipe subirrigation and drainage. The solution can be used to determine a flow net within the groundwater flow region and the associated water table shape. The solution is general. The effects of several hydraulic and geometrical parameters on the groundwater system, such as thickness of saturated zone, position of subirrigation and drainage pipes, heads in the subirrigation and drainage pipes, crop evapotranspiration, fraction of inflowing subirrigation water that exits through the drains, and the aquifer hydraulic conductivity system are evaluated. Calculations are presented showing how pipe spacing affects the shape of the water table. For example, with hydraulic conductivity of 10 m/d, evapotranspiration of 0.01 m/d, drainpipe radius of 0.05 m, and subirrigation pipe radius of 0.0375 m, calculations show that the maximum water table elevation for a pipe spacing of 40 m is 0.64 m greater than for a pipe spacing of 16 m when 40% of the input subirrigation water volume is being removed from the system by drainage. Finally, the general mathematical solution can be used to predict chemical movement as well as water flow through the system.

Disciplines

Agronomy and Crop Sciences | Hydrology | Soil Science | Tectonics and Structure

Comments

This article is published as Kirkham, Don, and Robert Horton. "The stream function of potential theory for a dual-pipe subirrigation-drainage system." Water resources research 28, no. 2 (1992): 373-387.Doi:10.1029/91WR02484 .Posted with permission.

WATER RESOURCES RESEARCH, VOL. 28, NO. 2, PAGES 373–387, FEBRUARY 1992

The Stream Function of Potential Theory for a Dual-Pipe Subirrigation-Drainage System

DON KIRKHAM AND ROBERT HORTON

Agronomy Department, Iowa State University, Ames

An exact mathematical solution to Laplace's equation is presented for appropriate boundary conditions associated with the problem of dual-pipe subirrigation and drainage. The solution can be used to determine a flow net within the groundwater flow region and the associated water table shape. The solution is general. The effects of several hydraulic and geometrical parameters on the groundwater system, such as thickness of saturated zone, position of subirrigation and drainage pipes, heads in the subirrigation and drainage pipes, crop evapotranspiration, fraction of inflowing subirrigation water that exits through the drains, and the aquifer hydraulic conductivity system are evaluated. Calculations are presented showing how pipe spacing affects the shape of the water table. For example, with hydraulic conductivity of 10 m/d, evapotranspiration of 0.01 m/d, drainpipe radius of 0.05 m, and subirrigation pipe radius of 0.0375 m, calculations show that the maximum water table elevation for a pipe spacing of 40 m is 0.64 m greater than for a pipe spacing of 16 m when 40% of the input subirrigation water volume is being removed from the system by drainage. Finally, the general mathematical solution can be used to predict chemical movement as well as water flow through the system.

1. INTRODUCTION

The flow problem of dual-pipe subirrigation and drainage is illustrated in Figures 1 and 2. The problem involves the use of the stream function to determine heads, velocities, and other hydrological quantities for a dual-pipe subirrigation-drainage system patented by *Thornton* [1985]. We already have done some analysis of a related problem that used rectangular (slit) tubes [*Kirkham and Horton*, 1986]. Here cylindrical (circle) tubes are used for both subirrigation water supply and subsurface drainage sinks.

Literature on subirrigation includes works by *Ernst* [1962], *Skaggs* [1981], *Brandyk and Wesseling* [1987], *Willardson* [1987], *Van Bakel* [1988], *Criddle and Kalisvaart* [1967], *Cooper and Fouss* [1988], and *Melvin et al.* [1990]. A main aim of this paper is to present formulas for the water table arch heights z and H as they depend on tube spacing, evapotranspiration coefficient, and other parameters of Figure 2. This paper identifies system parameters and shows how they influence subsurface flow and water table shapes that are useful for groundwater management and crop growth.

2. ANALYSIS

2.1. *The Stream Function*

Streamlines serve as a graphical representation of the stream function and are indicated in Figures 1 and 2. Some of the streamlines go to drainage outflow (a sink) and some to evapotranspiration (another sink). All streamlines originate at the irrigation tube (the source). Because Figure 1 has symmetry, we may consider flow only in the space OAD-D'EPO, less the space occupied by tubes and by a water table arch DD'ED. In our work it is to be understood that the flow region or problem domain extends a distance of 1 m perpendicular to the plane of the figure.

Paper number 91WR02484.
0043-1397/92/91WR-02484$05.00

We denote the stream function by ψ ((m^3 m^{-1}) d^{-1}) and indicate it graphically in Figure 2 as a set of arrowed lines $0.0\psi_0$, $0.2\psi_0$, $0.4\psi_0$, $\cdots$, $1.0\psi_0$. Here ψ_0 is given by the relation

$$\psi_0 = Q/2 \tag{1}$$

where $Q/2$ is the total flow per 1-m length of inflow tube moving to the left into the flow region.

In Figure 2 the streamline designated $0.4\psi_0$ is particularly important because we select in the figure the quantity $0.4\psi_0$ to be the amount of water ($0.4\psi_0 - 0.0\psi_0 = 0.4\psi_0$) that enters the drain tube sink. In general, we let f denote the fraction of flow going to the drain tube. Therefore an amount $(1 - f)\psi_0$ goes into evapotranspiration. The $0.4\psi_0$ streamline is branched at point P. One branch $0.4\psi_0$ moves up and another branch, also $0.4\psi_0$, moves down. The $0.4\psi_0$ and other streamlines are schematic.

We let e ((m^3 m^{-2}) d^{-1}) represent the evapotransportation per surface area assumed to be constant and take s to be the distance between an adjacent pair of tube centers. We can now write an extremely important continuity equation. It is

$$es = (1 - f)\psi_0 \tag{2}$$

2.2. *Evapotranspiration e, Water Table Arch z*

To make sure physically that e is a constant along the water table arch we assume, as in our earlier papers [*Kirkham and Horton*, 1986; *Kirkham and Powers*, 1984 (hereinafter KP), p. 113] that the flow in the water table arch (above line ED in Figure 2) is straightened vertically by infinitesimally thin, perfectly smooth, closely spaced, rigid, impervious, "metal" membranes. These membranes, when considered in adjacent pairs, serve as piezometers, which give the water table height z above the line ED. This height z in our analysis is an unknown to be determined as is the water table arch height H, which is a special value of z at the point D in the figure.

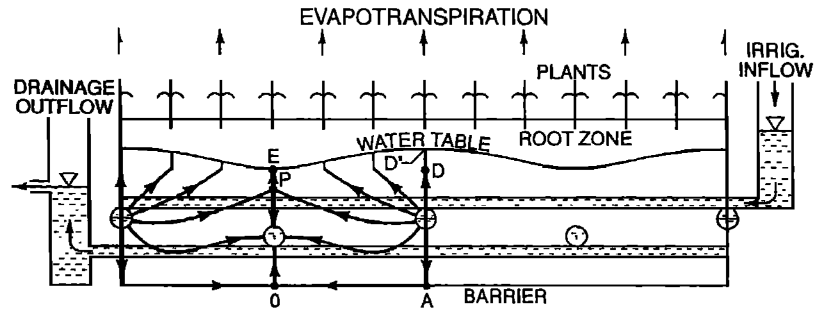

Fig. 1. The dual-pipe subirrigation-drainage system (the two horizontal pipes are outside the analysis domain).

In our analysis we initially assume that the porous medium (soil) in the arch region has been replaced by infinitely permeable "gravel" that makes adjacent metal strips true piezometers. Later we can replace the fictitious gravel by the original soil to get a correction factor [KP, p. 119]. For low water table heights our solution, z or H, either for gravel or for replaced soil, becomes exact.

2.3. *Laplace's Equation for the Stream Function*

We use Laplace's equation for the stream function to solve our problem. We solve the equation first for fictitous narrow rectangular vertical "slit tubes" (slits) that we temporarily imagine to replace the circle tubes (circles) in Figure 2. Later we shrink these slits to points which then become centers of (near) circle tubes, for which we get the potential function and desired heads. These near-circle tubes will be associated with actual drain tubes of finite diameter.

In Figure 2, but now with the imagined slits replacing the circle tubes, let us take the bottom of the left slit at $(0, \alpha)$, its center at $(0, \beta)$ and its top at $(0, \gamma)$. The corresponding values for the right slit are (s, a), (s, b), and (s, c). The slits are considered to be so thin that flow passing into and out of the top and bottom of a slit may be neglected.

That the stream function $\psi = \psi(x, y)$ ((m^3 m^{-1}) d^{-1}) satisfies Laplace's equation for porous media is shown in KP (p. 108). The equation for Cartesian coordinates is

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = 0 \tag{3}$$

The equation is applicable also when the potential function $\Phi = k\phi$ or just ϕ is used in place of the stream function ψ. This means that ψ may also represent the potential function, and if either ψ or $\partial\psi/\partial n$ (n is a normal distance) is prescribed on the boundary or if ψ is prescribed on part of the boundary and $\partial\psi/\partial n$ on the remaining part then there exists a unique solution ψ of Laplace's equation.

In listing our boundary condition (BC) equations, we use an index

$$m = 1, 2, \cdots \tag{4}$$

and boundary constants a_m, b_m, c_m that are associated with the top boundary ED, right boundary AD, and left boundary OE (Figure 2), respectively.

2.4. *Boundary Conditions for ψ for Slit Tubes*

With Figure 2 in view, we consider the shown circle tubes to be replaced by our fictitious slit tubes, and we now write the BCs for bottom, right, top, and left boundaries in terms of ψ for the slit tube system, as follows:

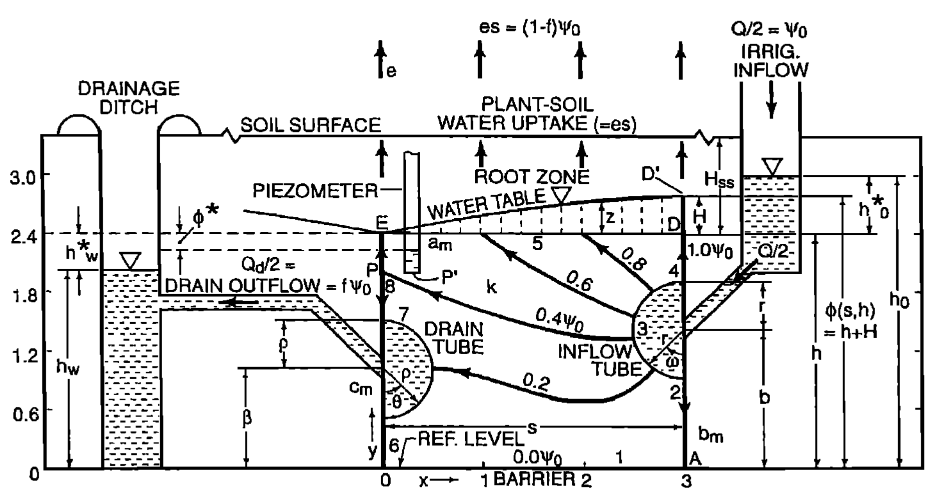

Fig. 2. Geometry, hydraulics, and most symbols for a typical flow semisection of the dual-pipe subirrigation-drainage system; k is hydraulic conductivity, e is evapotranspiration.

Bottom

(BC1) $\psi(x, 0) = 0 \qquad 0 \le x \le s$

Right b_m

(BC2) $\psi(s, y) = 0 \qquad 0 \le y \le a$

(BC3) $\psi(s, y) = \dfrac{y-a}{c-a}\psi_0 \qquad a \le y \le c$

(BC4) $\psi(s, y) = \psi_0 \qquad c \le y < h$

Top a_m

(BC5) $\psi(x, h) = f\psi_0 + (1-f)\dfrac{x}{s}\psi_0 \qquad 0 < x < s$

Left c_m

(BC6) $\psi(0, y) = 0 \qquad 0 \le y \le \alpha$

(BC7) $\psi(0, y) = \dfrac{y-\alpha}{\gamma-\alpha}f\psi_0 \qquad \alpha \le y \le \gamma$

(BC8) $\psi(0, y) = f\psi_0 \qquad \gamma \le y \le h$

To satisfy these BCs for slit tubes, except possibly in (BC5) for points $(0, h)$ and (s, h), which may for now be considered as removed from the flow medium, we select (see KP, especially pp. 99–101 substituting ψ for ϕ) the function ψ as

$$\frac{\psi}{\psi_0} = \sum a_m \sin\frac{m\pi x}{s}\frac{\sinh\dfrac{m\pi y}{s}}{\sinh\dfrac{m\pi h}{s}} + \sum b_m \sin\frac{m\pi y}{h}\frac{\sinh\dfrac{m\pi x}{h}}{\sinh\dfrac{m\pi s}{h}} + \sum c_m \sin\frac{m\pi y}{h}\frac{\sinh\dfrac{m\pi(s-x)}{h}}{\sinh\dfrac{m\pi s}{h}} \tag{5}$$

where the summing is for $m = 1, 2, \cdots, \infty$ (which is to be understood in the following for all infinite sums, unless otherwise stated).

In (5) we see, because $\sinh m\pi y/s = 0$ at $y = 0$, that (BC1) is satisfied for arbitrary values of a_m, b_m, and c_m. To satisfy the remaining BCs, (BC2)–(BC8), we use our a_m, b_m and c_m, and evaluate them by Fourier half-range sine series [KP, p. 67] to obtain

$$a_m = \frac{2}{\pi}\left(\frac{f}{m} - \frac{\cos m\pi}{m}\right) \tag{6}$$

$$b_m = \frac{2}{\pi}\left[\frac{1}{\pi}\frac{h}{c-a}\frac{1}{m^2}\left(\sin\frac{m\pi c}{h} - \sin\frac{m\pi a}{h}\right) - \frac{\cos m\pi}{m}\right] \tag{7}$$

$$c_m = \frac{2}{\pi}f\left[\frac{1}{\pi}\frac{h}{\gamma-\alpha}\frac{1}{m^2}\cdot\left(\sin\frac{m\pi\gamma}{h} - \sin\frac{m\pi\alpha}{h}\right) - \frac{\cos m\pi}{m}\right] \tag{8}$$

2.5. *Converting Slit Tubes to Circle Tubes*

2.5.1. *The a_m for circle tubes, shrinking the slits to points.* The a_m for circle tubes are the same as for slit tubes (equation (6)) because the dimensions α, β, γ and a, b, c, of the slits are not involved. The parameter f in (6) ties the a_m to the slit geometry of the a_m and b_m, as will be seen below.

To convert the b_m and c_m for slit tubes of (7) and (8) to circle tubes we shrink the strip length $c - a$ and $\gamma - \alpha$ to their center points b and β. These center points are line sinks with the lines being surrounded by equipotentials, nearly circular, of "radii" ρ and r, at angles θ and ω in Figure 2. Hydrologically, an equipotential can be considered as a perforated tube surrounded by a thin layer of infinitely conductive gravel. The angles θ and ω with apexes at points $(0, \beta)$ and (s, b) are useful in describing flow near the tubes. We take θ and ω each equal to 90° to match a theoretical equipotential to a tube.

2.5.2. *The b_m for circle tubes centered at height b.* To get the b_m for circle tubes we write (7) as

$$b_m = \frac{2}{\pi}\left(p_m - \frac{\cos m\pi}{m}\right) \tag{9}$$

where p_m is defined by

$$p_m = \frac{1}{\pi}\frac{h}{c-a}\frac{1}{m^2}\left(\sin\frac{m\pi c}{h} - \sin\frac{m\pi a}{h}\right) \tag{10}$$

The height b of the right slit center above the barrier in Figure 2 is

$$b = [\tfrac{1}{2}(c-a)] + a \tag{11}$$

which gives b as (right slit)

$$b = \tfrac{1}{2}(c+a) \tag{12}$$

Dwight [1961, equation (401.09)] gives the identity (for any a or c)

$$\sin c - \sin a = 2\sin[\tfrac{1}{2}(c-a)]\cos[\tfrac{1}{2}(c+a)] \tag{13}$$

and (13), when applied to (10), gives (10) as

$$p_m = \frac{1}{\pi}\frac{h}{c-a}\frac{1}{m^2}2\sin\frac{m\pi(c-a)}{2h}\cos\frac{m\pi(c+a)}{2h} \tag{14}$$

Combining (12) and (14) gives

$$p_m = \frac{1}{\pi}\frac{h}{c-a}\frac{1}{m^2}2\sin\frac{m\pi(c-a)}{2h}\cos\frac{m\pi b}{h} \tag{15}$$

In (5) we see as from *Dwight* [1961, equations (48.002), (601.01)] and *Kaplan* [1959, chapter 6] that the particular sum involving b_m described by (9) and (15) converges. Therefore for a given desired accuracy the summation can be terminated with a given largest value of m; and $(c - a)$ can become as small as desired by shrinking $(c - a)$ to zero. Therefore, because by *Dwight* [1961, equation (415.01)] we have $\sin x = x$ when x is sufficiently small, we may write the sin term in (15) as

$$\sin \frac{m\pi(c-a)}{2h} = \frac{m\pi(c-a)}{2h} \tag{16}$$

and use (16) and (15) to get

$$p_m = \frac{1}{\pi}\frac{h}{c-a}\frac{1}{m^2}\left[2\,\frac{m\pi(c-a)}{2h}\cos\frac{m\pi b}{h}\right] \tag{17}$$

and (17) with cancelings is seen to give

$$p_m = \frac{1}{m}\cos\frac{m\pi b}{h} \tag{18}$$

so that (18), when used in (9), gives (for circles)

$$b_m = \frac{2}{\pi}\left(\frac{1}{m}\cos\frac{m\pi b}{h} - \frac{1}{m}\cos m\pi\right) \qquad 0 < b < h \tag{19}$$

Equation (19) is our sought result for the b_m for circle tubes.

2.5.3. *The c_m for circle tubes.* As (7) gives (19) so (8) gives the equation for the constants c_m for circular tubes (with use of the relation $\frac{1}{2}(\gamma + \alpha) = \beta$, obtained as in (12)):

$$c_m = \frac{2}{\pi}f\left(\frac{1}{m}\cos\frac{m\pi\beta}{h} - \frac{1}{m}\cos m\pi\right) \qquad 0 < \beta < h \tag{20}$$

With ψ the stream function of (5) now determined, and in view of (6), (19), and (20) we next use ψ to get the velocity potential Φ, which is defined by $\Phi = k\phi$ (k is hydraulic conductivity). We will later develop formulas for the hydraulic head ϕ that will yield values of z and H as described in the introduction. With Φ at hand we can get velocities from $-k\partial\phi/\partial x$ and $-k\partial\phi/\partial y$ and then we can determine travel times of chemical movements, as in the work by *Kirkham and Affleck* [1977].

2.6. *The Velocity Potential Φ for Circle Tubes, Converting ψ to Φ*

The constants a_m, b_m and c_m of (6), (19), and (20) when substituted in (5), do satisfy the BCs of Figure 2 for a line source (s, b) and sink $(0, \beta)$ and, hence, for the circle tubes for the normalized stream function ψ/ψ_0. To get the needed heads and velocities for the flow region, we now convert ψ of (5) to a velocity potential Φ defined by

$$\Phi = k\phi \tag{21}$$

where k (m/d) is the hydraulic conductivity, assumed a constant of the flow medium, and ϕ (m) is the hydraulic or piezometric head. We choose the reference level of ϕ to be at $y = 0$. To convert ψ to Φ we must keep the physics straight. Figure 2 shows a piezometer. Water stands in it to a certain height, the piezometric or standpipe water level. The height of this level above the reference level is a measure of ϕ or $\phi(x, y)$ at point P′ or P′(x, y) (ϕ^* in Figure 2 will be discussed later) shown at the base of the piezometer. It is this height ϕ (equal to the ratio Φ/k) that we now seek for all points in the flow medium, by converting ψ to Φ.

To convert ψ to Φ we use Cauchy-Riemann (CR) relations, which are

$$\partial\Phi/\partial x = \partial\psi/\partial y; \qquad \partial\Phi/\partial y = -\partial\psi/\partial x \tag{22}$$

These CR differential equations are derived and tabulated in some integrated forms in KP (especially pp. 105–106, 491–494).

To get Φ of (21) and (22) from ψ of (5) with the a_m, b_m, and c_m, as in (6), (19), and (20) and Φ subject to (22) we proceed as follows.

We define a variable $V(x, y)$, which we write by selecting Φ elements in Table 3-1 of KP from ψ elements in the right-hand side of (5). The form of $V(x, y)$ that we find for Φ is (independently of whether we use the a_m, b_m, and c_m of (6), (7), and (8) for slits or use (6), (19), and (20) for circles because points for centers of circles are special cases of the slits) given by (keeping (5) and Table 3-1 of KP in view) the expression

$$\begin{aligned} V(x, y) = {} & \sum a_m(-1)\cos\frac{m\pi x}{s}\,\frac{\cosh\dfrac{m\pi y}{s}}{\sinh\dfrac{m\pi h}{s}} \\ & + \sum b_m\cos\frac{m\pi y}{h}\,\frac{\cosh\dfrac{m\pi x}{h}}{\sinh\dfrac{m\pi s}{h}} \\ & + \sum c_m(-1)\cos\frac{m\pi y}{h}\,\frac{\cosh\dfrac{m\pi(s-x)}{h}}{\sinh\dfrac{m\pi s}{h}} \end{aligned} \tag{23}$$

With (23) now at hand we consider (5) as having been multiplied through by ψ_0 and write a general integrated form of the Φ of (22) as

$$\Phi(x, y) = \psi_0 V(x, y) + G \tag{24}$$

Here G is a constant. One may verify directly, by differentiation of (5) and (24), that ψ and Φ satisfy the CR conditions (22). In (24) Φ, ψ_0, and G have dimensions square meters per day; the function $V(x, y)$ is dimensionless.

To get G, first substitute (21) into (24), to get

$$k\phi(x, y) = \psi_0 V(x, y) + G \tag{25a}$$

Then in Figure 2 note that the water table by definition is at atmospheric pressure, and this is true in particular for the point $(0, h)$, where h is the height above the barrier that is also the reference level. Therefore we have $\phi(0, h) = h$ because pressure head in ϕ is zero and only gravitational head h remains (see KP, pp. 98–99); (25a) becomes

$$kh = \psi_0 V(0, h) + G \tag{25b}$$

which gives (assuming $V(0, h)$ is calculable) the relation

$$G = kh - \psi_0 V(0, h) \tag{25c}$$

We put G as given by (25c) in (25a), divide the result by k and find after rearranging

$$\phi(x, y) - h = (\psi_0/k)[V(x, y) - V(0, h)] \qquad (25d)$$

which alternatively gives

$$\phi(x, y) = h + (\psi_0/k)[V(x, y) - V(0, h)] \qquad (26a)$$

Equation (26a) does not involve the evapotranspiration constant e. We return to (2) and divide both sides by the hydraulic conductivity k to find (2) as

$$\psi_0/k = (e/k)[s/(1-f)] \qquad (26b)$$

We put the right-hand side of (26b) in (25d) and get (we use from now on the a_m, b_m, and c_m for circles of (6), (19), and (20))

$$\phi(x, y) - h = \frac{e}{k}\frac{s}{1-f}[V(x, y) - V(0, h)], \qquad (26c)$$

which is a key equation involving e. In (26c), in the V terms, a number of parameters do not appear. A complete form of $V(x, y)$ is $V(x, y; f, s, h, \beta, b)$ as is seen from (23) and (6), (19), and (20). The expanded form of V does not include tube radii parameters ρ and r. We account for tube radii by identifying equipotential circles of radii ρ and r that encircle points $(0, \beta)$ and (s, b) with actual gravel-surrounded pipes of radii ρ and r at $(0, \beta)$ and (s, b). $V(x, y)$ in (26c) is the most important function of this paper.

2.7. *Formulas for Heads z and H and Star Notation*

From (26c) we can easily get the water table arch heights z and H, as we recall them from Figure 2. We put $y = h$ in (26c) and recognize that z must be given by

$$z = \phi(x, h) - h \qquad (26c')$$

We then get z as (change y to h in (26c) and use (26c')

$$z = \frac{e}{k}\frac{s}{1-f}[V(x, h) - V(0, h)] \qquad (26d)$$

For H, change x to s in (26d), and, in view of Figure 2, change z to H and determine H (assuming the quantity $[V(s, h) - V(0, h)]$ is calculable) as

$$H = \frac{e}{k}\frac{s}{1-f}[V(s, h) - V(0, h)] \qquad (26e)$$

Some more heads, h_w and h_0, shown in Figure 2, can be derived from (26c). The head h_w is for drainage ditch or sink (well) head, and the head h_0 is for irrigation inflow or (originating) head. For h_w we put $x = \rho$ and $y = \beta$ in (26c) (making $\theta = 90°$ and $\omega = 90°$ as shown above (9)) and get $\phi(\rho, \beta)$ as

$$\phi(\rho, \beta) - h = \frac{e}{k}\frac{s}{1-f}[V(\rho, \beta) - V(0, h)] \qquad (26f)$$

But h_w (see Figure 2) is just another notation for $\phi(\rho, \beta)$. Therefore we change $\phi(\rho, \beta)$ in the last equation to h_w and find (26f) as

$$h_w - h = \frac{e}{k}\frac{s}{1-f}[V(\rho, \beta) - V(0, h)] \qquad (26g)$$

and similarly we put $x = s - r$ and $y = b$ in (26c) and use $\phi(s - r, b) = h_0$ to get

$$h_0 - h = \frac{e}{k}\frac{s}{1-f}[V(s - r, b) - V(0, h)] \qquad (26h)$$

The left-hand sides of (26c), (26g), and (26h) may be abbreviated as

$$\phi^*(x, y) = \phi(x, y) - h \qquad (26i)$$

$$h_w^* = h_w - h \qquad (26j)$$

$$h_0^* = h_0 - h \qquad (26k)$$

and these three starred heads are for the flow region points (x, y), (ρ, β), and $(s - r, b)$, each with respect to the level $y = h$. Equations (26i)–(26k) are definitions of ϕ^*, h_w^*, and h_0^*. The heights z and H of the water table arch of Figure 2 can be obtained in terms of starred values. We put $y = h$ in (26i), use (26c') and get

$$\phi^*(x, h) = z, \qquad (26l)$$

which for the maximum value H of z becomes

$$\phi^*(s, h) = H. \qquad (26m)$$

In the right-hand side of (26a) and (26m) we may use (26d) and (26e). We shall next work on $V(x, y)$.

2.8. *An Expanded Form of V(x, y) for Rapid Convergence*

There are difficulties in working with (23). For example, the first sum is slowly converging near $y = h$, the second near $x = s$, and the third near $x = 0$. Also, the first sum is nonconvergent at $(0, h)$, the second at (s, h), and the third at $(0, h)$. We can (see microfiche supplement[1]) change [after *Kirkham*, 1958] the right-hand side of (23) to a rapidly converging form to get

$$V(x, y) = (1-f)\left(\frac{h-y}{s} - \frac{\ln 2}{\pi}\right) + \frac{1}{\pi} f \ln \frac{\cosh \dfrac{\pi(h-y)}{s} - \cos \dfrac{\pi x}{s}}{\cosh \dfrac{\pi x}{h} + \cos \dfrac{\pi y}{h}} + \frac{1}{\pi} \ln \frac{\cosh \dfrac{\pi(s-x)}{h} + \cos \dfrac{\pi y}{h}}{\cosh \dfrac{\pi(h-y)}{s} + \cos \dfrac{\pi x}{s}} + \frac{1}{2\pi} \ln \frac{A^f}{B} + T_2 + T_4 + T_6, \qquad (27a)$$

where

[1] The supplementary material is available with entire article on microfiche. Order from American Geophysical Union, 2000 Florida Avenue, N.W., Washington, DC 20009. Document W91-007; \$2.50. Payment must accompany order. Also available from Photo Service, Iowa State University, Ames, Iowa 50011. Request supplement to publication STP 5/22/91; enclose \$0.50 per copy.

$$A = \left[\cosh\frac{\pi x}{h} - \cos\frac{\pi(\beta + y)}{h}\right]\left[\cosh\frac{\pi x}{h} - \cos\frac{\pi(\beta - y)}{h}\right] \tag{27b}$$

$$B = \left[\cosh\frac{\pi(s - x)}{h} - \cos\frac{\pi(b + y)}{h}\right] \cdot \left[\cosh\frac{\pi(s - x)}{h} - \cos\frac{\pi(b - y)}{h}\right] \tag{27c}$$

$$T_2 = \sum \frac{2}{\pi}\frac{1}{m}(f - \cos m\pi)(-1)\cos\frac{m\pi x}{s} \cdot \exp\left(-\frac{m\pi h}{s}\right)[\cosh m\pi(h-y)/s]/[\sinh m\pi h/s] \tag{27d}$$

$$T_4 = \sum \frac{2}{\pi}\frac{1}{m}\left(\cos\frac{m\pi b}{h} - \cos m\pi\right)\cos\frac{m\pi y}{h} \cdot \exp\left(-\frac{m\pi s}{h}\right)[\cosh m\pi(s-x)/h]/[\sinh m\pi s/h] \tag{27e}$$

$$T_6 = \sum \frac{2}{\pi} f \frac{1}{m}\left(\cos\frac{m\pi\beta}{h} - \cos m\pi\right)(-1)\cos\frac{m\pi y}{h} \cdot \exp\left(-\frac{m\pi s}{h}\right)[\cosh m\pi x/h]/[\sinh m\pi s/h] \tag{27f}$$

The removed points $(0, h)$ and (s, h) of boundary condition (BC5) (also see above (5)) need further consideration.

2.9. *Expressions of Indeterminate Forms*

Equation (27*a*) has purposely been put into a form to remove expressions which are of indeterminate forms at $(0, h)$ and (s, h) (desired infinities occur at the drain sink $(0, \beta)$ and the irrigation source (s, b)). The indeterminate form at point $(0, h)$ occurs in the second term of the right-hand side of (27*a*) as

$$\frac{1}{\pi} f \ln\frac{1-1}{1-1} = \frac{1}{\pi} f[-\infty - (-\infty)] \tag{28a}$$

and the indeterminate form at (s, h) occurs in the third term of the right-hand side of (27*a*) as

$$\frac{1}{\pi}\ln\frac{1-1}{1-1} = \frac{1}{\pi}[-\infty - (-\infty)] \tag{28b}$$

To deal with (28*a*) and (28*b*) we first return to (27*a*) and change it to

$$V(x, y) = V_a + V_b + V_c + V_d + T_2 + T_4 + T_6 \tag{29a}$$

where

$$V_a(x, y) = (1 - f)\left(\frac{h - y}{s} - \frac{\ln 2}{\pi}\right) \tag{29b}$$

$$V_b(x, y) = \frac{1}{\pi} f \ln \frac{\cosh\frac{\pi(h-y)}{s} - \cos\frac{\pi x}{s}}{\cosh\frac{\pi x}{h} + \cos\frac{\pi y}{h}} \tag{29c}$$

$(0, h)$ excluded; use (36)

$$V_c(x, y) = \frac{1}{\pi}\ln\frac{\cosh\frac{\pi(s-x)}{h} + \cos\frac{\pi y}{h}}{\cosh\frac{\pi(h-y)}{s} + \cos\frac{\pi x}{s}} \tag{29d}$$

(s, h) excluded; use (37)

$$V_d(x, y) = \frac{1}{2\pi}\ln\frac{A^f}{B} \tag{29e}$$

with A and B as in (27*b*) and (27*c*),

$$T_2(x, y) = \text{right-hand side of (27}d) \tag{29f}$$

$$T_4(x, y) = \text{right-hand side of (27}e) \tag{29g}$$

$$T_6(x, y) = \text{right-hand side of (27}f) \tag{29h}$$

For $(0, h)$ we replace in (29*c*) x and y by ε and $h - \delta$ (where ε and δ are small quantities) and, after using the identity

$$\cos \pi(h - \delta)/h = -\cos \pi\delta/h \tag{30}$$

we find (29*c*) as

$$V_b(\varepsilon, h - \delta) = \frac{1}{\pi} f \ln\frac{\cosh\frac{\pi\delta}{s} - \cos\frac{\pi\varepsilon}{s}}{\cosh\frac{\pi\varepsilon}{h} - \cos\frac{\pi\delta}{h}} \tag{31}$$

But from *Dwight* [1961, equations (415.02), (657.2)] we have

$$\cosh A - \cos B = \left(1 + \frac{A^2}{2!} + \cdots\right) - \left(1 - \frac{B^2}{2!} + \cdots\right) \tag{32}$$

so that, when A and B are small enough, (32) becomes (neglect terms of order $A^4, B^4, A^6, B^6, \cdots$)

$$\cosh A - \cos B = \tfrac{1}{2}(A^2 + B^2) \tag{33}$$

We apply (33) to (31) and find (31) after some simplification as

$$V_b(\varepsilon, h - \delta) = \frac{1}{\pi} f \ln\left(\frac{h^2}{s^2}\frac{\delta^2 + \varepsilon^2}{\varepsilon^2 + \delta^2}\right) \tag{34}$$

This last equation, for ε and $\delta \to 0$, may be written as

$$V_b(0, h) = \frac{1}{\pi} f \ln\frac{h^2}{s^2} \tag{35}$$

or

$$V_b(0, h) = \frac{2}{\pi} f \ln\frac{h}{s} \tag{36}$$

which takes care of point $(0, h)$. In (34), $\delta^2 + \varepsilon^2 = (\text{radius})^2$.

As we found $V_b(0, h)$ of (36) from (29c), we can similarly find $V_c(s, h)$ from (29d). We replace, in (29d), x by $s - \varepsilon$ and y by $(h - \delta)$ and after using the identity (30) we find (29c) as

$$V_c(s, h) = \frac{2}{\pi} \ln \frac{s}{h} \tag{37}$$

3. FURTHER ANALYSIS

With (36) and (37) at hand we can now readily get some special values of $V(x, y)$ of (29a). We start with $V(0, h)$.

3.1. *Equations for V(0, h)*

We return to (29a), insert $x = 0$ and $y = h$ and get

$$V(0, h) = V_a(0, h) + V_b(0, h) + \cdots + T_6(0, h) \tag{38}$$

In (29b) we insert $x = 0$ and $y = h$, rearrange, and find $V_a(0, h)$ as

$$V_a(0, h) = (-1)(1 - f)\frac{\ln 2}{\pi} \tag{39}$$

From (36) we copy $V_b(0, h)$ as

$$V_b(0, h) = \frac{2}{\pi} f \ln \frac{h}{s} \tag{40}$$

in (29d) we insert, in each side, $x = 0$ and $y = h$, to find with the aid of *Dwight* [1961, equation (652.5)]

$$V_c(0, h) = \frac{2}{\pi} \ln \sinh \frac{\pi s}{2h} \tag{41}$$

In (29e) we insert, in each side, $x = 0$, $y = h$, and with the aid of *Dwight* [1961, equations (404.22), (401.03), (401.04)] we find (29e) as

$$V_d(0, h) = \frac{1}{\pi} \ln \frac{\left(2 \cos \dfrac{\pi\beta}{2h}\right)^f}{\cosh \dfrac{\pi s}{h} + \cos \dfrac{\pi b}{h}} \tag{42}$$

For the T terms in (38) we return to (27d)–(27f), insert there $(x, y) = (0, h)$ and partly simplify to find T_2, T_4 and T_6 as

$$T_2(0, h) = \sum \frac{2}{\pi}\frac{1}{m}(f - \cos m\pi)(-1)(1) \cdot \exp\left(-\frac{m\pi h}{s}\right)\frac{1}{\sinh \dfrac{m\pi h}{s}} \tag{43}$$

$$T_4(0, h) = \sum \frac{2}{\pi}\frac{1}{m}\left(\cos\frac{m\pi b}{h} - \cos m\pi\right)(\cos m\pi) \cdot \exp\left(-\frac{m\pi s}{h}\right)\frac{\cosh \dfrac{m\pi s}{h}}{\sinh \dfrac{m\pi s}{h}} \tag{44}$$

$$T_6(0, h) = \sum \frac{2}{\pi} f \frac{1}{m}\left(\cos\frac{m\pi\beta}{h} - \cos m\pi\right)(-1)\cos m\pi \cdot \exp\left(-\frac{m\pi s}{h}\right)\frac{1}{\sinh \dfrac{m\pi s}{h}} \tag{45}$$

We insert the right-hand side of (39)–(45) in (38) and find $V(0, h)$ as

$$V(0, h) = (-1)(1 - f)\frac{\ln 2}{\pi} + [\text{right-hand sides of (40)–(45)}] \tag{46}$$

3.2. *Equations for V(s, h), V(ρ, β), and V(s − r, b), Tube Radii*

As we found $V(0, h)$ of (38)–(46) we may find $V(s, h)$, $V(\rho, \beta)$, and $V(s - r, b)$. The latter two V expressions bring the tube radii ρ and r into the analysis. For now, we abbreviate $V(s, h)$ and $V(\rho, \beta)$ and $V(s - r, b)$ as

$$V(s, h) = [\text{right-hand sides of (29b)–(29h) at } (s, h), \text{ and (38)}] \tag{47}$$

$$V(\rho, \beta) = [\text{right-hand sides of (29b)–(29h) at } (\rho, \beta)] \tag{48}$$

$$V(s - r, b) = [\text{right-hand sides of (29b)–(29h) at } (s - r, b)] \tag{49a}$$

We next complete (47)–(49a) in terms of the original parameters. Some reference formulas are [*Dwight*, 1961, equations (652.3), (404.22), (654.5)]

$$\cosh A - 1 = 2 \sinh^2 \frac{A}{2} \tag{49b}$$

$$1 + \cos A = 2 \cos^2 \frac{A}{2} \tag{49c}$$

$$\frac{\cosh A}{\sinh A} = \coth A \tag{49d}$$

$$e^{-A}/\sinh A = \coth A - 1 \tag{49e}$$

With (29a)–(49e) one can determine $V(x, y)$. See the microfiche supplement to find the assembled working equation for efficient computation of $V(x, y)$ for all points including corners of the flow region.

4. FORMULA FOR [V(s, h) − V(0, h)] (FOR H OF (26e))

From (38)–(47) with manipulation (using (49b)–(49e) and *Dwight* [1961, equations (403.4), (652.12)]),

$$V(s, h) - V(0, h) = \frac{2}{\pi}(1+f)\ln\frac{(s/h)}{\sinh\dfrac{\pi s}{2h}}$$

$$+ \frac{f}{\pi}\ln\frac{\cosh\dfrac{\pi s}{h} + \cos\dfrac{\pi\beta}{h}}{2\cos^2(\pi\beta/2h)} + \frac{1}{\pi}\ln\frac{\cosh\dfrac{\pi s}{h} + \cos\dfrac{\pi b}{h}}{2\cos^2(\pi b/2h)}$$

$$+ \sum\left(\sin^2\frac{m\pi}{2}\right)\frac{4}{\pi}\frac{1}{m}(1+f)\left(\coth\frac{m\pi h}{s} - 1\right)$$

$$+ \sum\frac{4}{\pi}\frac{1}{m}g_m \exp\left(-\frac{m\pi s}{h}\right)\tanh\frac{m\pi s}{2h} \qquad (50)$$

where

$$g_m = \cos^2\frac{m\pi b}{2h} + f\cos^2\frac{m\pi\beta}{2h} \qquad m = 1, 3, \cdots$$

$$g_m = \sin^2\frac{m\pi b}{2h} + f\sin^2\frac{m\pi\beta}{2h} \qquad m = 2, 4, \cdots$$

for $(0 < \beta < h;\ 0 < b < h)$.

5. RESULTS AND DISCUSSION

5.1. *Knowns and Unknowns*

To present results we need to have "knowns" and "unknowns" (independent and dependent variables) in mind. We take them, given in Figure 2, as

knowns: $x, y, e, k, f, h, \beta, b, \rho, r$ (51*a*)

unknowns: $Q/2,\ \phi(x, y),\ \phi^*(x, y),\ z,\ H,\ h_w^*,\ h_0^*$ (51*b*)

Also we need as "unknowns" a flow net; a set of values of $\phi(x, y)$ for the flow net; a set of graphs of z versus x, with k as parameter; and graphs of h_w^*, h_0^* and H all versus s.

We consider the "unknowns" in order; they are $Q/2$ and heads.

5.2. *Formulas for Q/2 and Heads, a Flow Net, Etc.*

For $Q/2$, from (1) and (2) we find

$$Q/2 = (es)/(1-f) \qquad (52)$$

Note that $Q/2$ appears to depend on e, s, and f only (not on k, not on h, β, b, ρ, or r); but a certain $\phi(x, y)$ distribution implied by the f in (52) is needed to get the $Q/2$ of (52); this $\phi(x, y)$ with special values is our next unknown.

For $\phi(x, y)$, from (26*c*) (with $V(x, y)$ as in (29*a*)–(29*h*), and $V(0, h)$ as in (38)–(45)) we find

$$\phi = h + [(e/k)s/(1-f)][V(x, y) - V(0, h)] \qquad (53a)$$

For $\phi^*(x, y)$, by definition (or (26*i*)) we have

$$\phi^*(x, y) = \phi - h \qquad (53b)$$

For z, from (26*d*) (with $V(x, y)$ obtained by inserting h for y in (29*a*)–(29*h*) and with $V(0, h)$ as in (38)–(45)), we find

$$z = [(e/k)s/(1-f)][V(x, h) - V(0, h)] \qquad (54)$$

For H, from (26*e*) and (50) we find

$$H = [(e/k)s/(1-f)][V(s, h) - V(0, h)] \qquad (55)$$

We make two remarks about H of (55).

1. If, in Figure 2, the distance H_{ss} is less than H, there will be a "blowout" or seep above the inflow tube. Blowouts are unlikely to occur when (es/k) is small.

2. Drain radii ρ and r of the "knowns" (51*a*) do not appear explicitly in H of (55) (nor in (52), (53*a*), or (54)). This is because we assumed line sources and sinks of zero radii at the outset of our derivation, and line sources and sinks do not have radii. Nevertheless, our problem assumes (51*a*) the tube radii are nonzero; otherwise, no water would flow through the flow region and our problem would not exist. The larger the radii of the tubes, the less head difference will be required to get a certain amount $Q/2(= es/(1-f)$ of (52)) of flow through the flow region. The reason for this is that large tubes cut down convergence loss of head in the flow region. Analytically, H is connected to ϕ by $H = \phi(s, h) - h$.

5.3. *The Head H*

Because H of (55) is so important we expand it from (55) and (50) as

$$H = \frac{e}{k}\frac{s}{1-f}(A + B + C + D + E) \qquad (0 \le f < 1) \qquad (56a)$$

where

$$A = \frac{2}{\pi}(1+f)\ln\frac{s/h}{\sinh\dfrac{\pi s}{2h}} \qquad (56b)$$

$$B = \frac{f}{\pi}\ln\frac{\cosh\dfrac{\pi s}{h} + \cos\dfrac{\pi\beta}{h}}{2\cos^2(\pi\beta/2h)} + \frac{1}{\pi}\ln\frac{\cosh\dfrac{\pi s}{h} + \cos\dfrac{\pi b}{h}}{2\cos^2(\pi b/2h)} \qquad (56c)$$

$$C = \sum_{m=1,3,5,\cdots}\frac{4}{\pi}\frac{1}{m}(1+f)\left[\left(\coth\frac{m\pi h}{s}\right) - 1\right] \qquad (56d)$$

$$D = \sum_{m=1,3,5,\cdots}\frac{4}{\pi}\frac{1}{m}\left(\cos^2\frac{m\pi b}{2h} + f\cos^2\frac{m\pi\beta}{2h}\right) \cdot \exp\left(-\frac{m\pi s}{h}\right)\tanh\frac{m\pi s}{2h} \qquad (56e)$$

$$E = \sum_{m=2,4,6,\cdots}\frac{4}{\pi}\frac{1}{m}\left(\sin^2\frac{m\pi b}{2h} + f\sin^2\frac{m\pi\beta}{2h}\right) \cdot \exp\left(-\frac{m\pi s}{h}\right)\tanh\frac{m\pi s}{2h} \qquad (56f)$$

In (56*a*), f may not be 1 because of the factor $1/(1-f)$; and neither β nor b may be equal to h because of $2\cos^2(\pi\beta/2h)$ and $2\cos^2(\pi b/2h)$ in (56*c*).

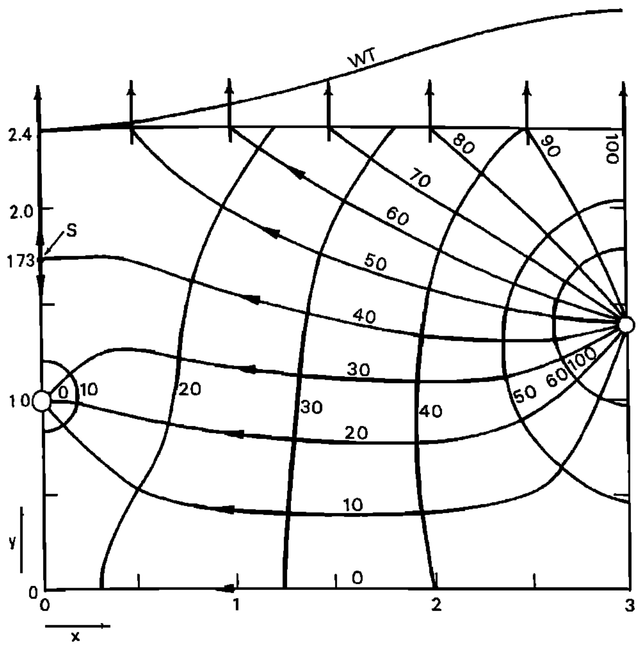

Fig. 3. A computed flow net with water table WT curve for $f = 0.4$, $s = 3$ m, $h = 2.4$, $\beta = 1.0$, $b = 1.4$, $\rho = 0.05$, $r = 0.0375$, and $e/k = 0.2$; maximum height of the WT arch is 0.6286 m.

The sums C, D, and E of (56a) should be summed to as many terms as are needed for a desired accuracy. We have taken enough terms in our calculations so that the number in the fourth place of decimals did not change when more terms were taken. To determine the number of terms to take for a particular sum see *Kaplan* [1959].

5.4. *The Heads h_w^* and h_0^**

We come now to h_w^* and h_0^* of our "unknowns." (Remember the asterisks denote heads with respect to level $y = h$ in Figure 2).

For h_w^*, from (26g) and (26j) (with $V(\rho, \beta)$ as in (29a)–(29h) and $V(0, h)$ as in (38)–(45)) we write h_w^* as

$$h_w^* = \frac{e}{k}\frac{s}{1-f}[V(\rho, \beta) - V(0, h)] \tag{57}$$

Similarly from (26h) (with $V(s - r, b)$ as in (29a)–(29h) and $V(0, h)$ as in (38)–(45)), we write h_0^* as

$$h_0^* = \frac{e}{k}\frac{s}{1-f}[V(s - r, b) - V(0, h)] \tag{58}$$

A main point about (57) and (58) is that, unlike heads z, H, ϕ, and ϕ^*, which are not functions of ρ and r, the head h_w^* is a function of ρ, and the head h_0^* is a function of r.

5.5. *A Flow Net*

A flow net (Figure 3) is our next "unknown." It consists of normalized equipotentials and streamlines graphed from ψ of (1), (5), (19) and (20) and Φ of (21) and (26c) for the parameters

$$e/k = 0.20, \quad f = 0.40, \quad s = 3\ (\text{m}), \quad h = 2.4\ (\text{m}),$$

$$(Q/2)/k = 1\ (\text{m}), \quad \beta = 1.0\ (\text{m}), \tag{59a}$$

$$b = 1.4\ (\text{m}), \quad \rho = 0.05\ (\text{m}), \quad r = 0.0375\ (\text{m})$$

In the net, streamlines and equipotentials intersect at 90°, as they should. The 40% ($f = 0.4$) streamline is important. It separates flow that goes into the drain tube from flow that goes into evapotranspiration. The 40% line is branched. It branches at a stagnation point labeled S at (0, 1.73). One branch goes to the top left corner of the flow region, and the other branch goes down the left side of the flow region to the drain tube. The 40% streamline approaches the left side of the flow region at the point S at 90°. This is unlike the approach of a guessed line $0.4\psi_0$ (not at 90°) in Figure 2. In Figure 3 the net is valid for $e/k = 0.20$, but it also is valid (Appendix A) for other values of e/k because the net is normalized and the ratio e/k cancels in the normalization process. At the top of the figure a water table WT curve, z versus x, of (54) is shown for the parameters of (59a). The highest point of the water table curve is not labeled. Its value is $H = 0.6286$ m. The curve is flat at $x = 0$ and again at $x = s$. It must be, by symmetry. The flatness property is useful in using a spline to connect plotted points, especially if only the two head points, $\phi^*(0, h)$ and $\phi^*(s, h)$, of a water table curve are known.

5.6. *A Set of $\phi(x, y)$: $V(x, y)$, $\phi^*(x, y)$*

Our next "unknown" is a set of values of $\phi(x, y)$ that we used for the flow net, and Figure 4 contains mesh points, 0.6 m apart, with associated numerical values, triplets, $V(x, y)$, $\phi^*(x, y)$, and $\phi(x, y)$, used in getting the equipotentials of the flow net. For example, at the point at top right (3, 2.4) we have $V(x, y) = 0.2174$, $\phi^* = 0.6286$, and $\phi = 3.0286$. In line 2 of Figure 4 the values of $\phi^*(x, h)$ are of particular importance because by (26l), namely $\phi^*(x, h) = z$, we may write from line 2, the points (x, y), where $z = y - 2.4$, as

$$(x, z) = (0, 0.0000), (0.6, 0.0463),$$

$$(1.2, 0.1680), \cdots, (3.0, 0.6286), \tag{59b}$$

for the WT curve of Figure 3. To check the triplets in Figure 4 we may (because $(e/k)s/(1 - f) = 1.0000$, $h = 2.4$, and $V(0, h) = -0.4112$) derive from (38) and (53b) the relation $V + \phi - 2\phi^* = 1.9888$; that is, in the lower left of Figure 4 the point (0, 0.0) gives $2.5596 + (-0.2516) - 2(0.1596) = 1.9888$ (see Appendix B).

In Figure 4, at top left, a piezometer is shown with its bottom at the point (0, 1.8). Water stands in the piezometer to a height $\phi(0, 1.8) = 2.4695$ m above the barrier. The piezometer water level also is shown as $\phi^*(0.0, 1.8) = 0.0695$ m. The water is moving upward at point (0.0, 1.8).

5.7. *Graphs of Water Arch Height z*

The next "unknown" below (51b) is a set of graphs of z versus x with k as a parameter, and Figure 5 gives such a set for selected values of k = 0.025, 0.050, 0.100 and 1.00 (m/d), where the other parameters (the same for each curve) are e = 0.01 (m/d), f = 0.4, s = 3.0 (m), h = 2.4 (m), β = 1.0 (m), b = 1.4 (m) (and with ρ and r being any small values such as ρ = 0.05 (m) and r = 0.0375 (m), and their corresponding heads that can be identified with the tube). Points z versus x from which the curves of Figure 5 were

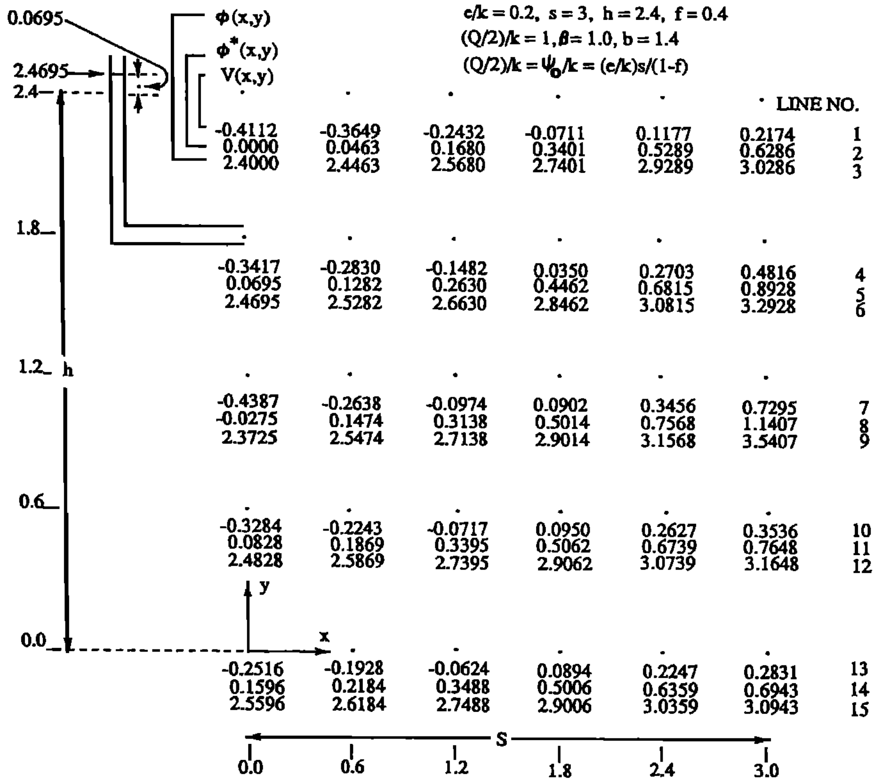

Fig. 4. Triplets of heads $\phi(x, y)$ and $\phi^*(x, y)$ and function $V(x, y)$ at grid points 0.6 m apart for a flow cross section with parameters (e/k and f dimensionless, others in meters) as at top right. The figure has a piezometer with end at (0, 1.8) with water standing in it at $\phi(0, 1.8) = 2.4695$ m (above $y = 0$) and at $\phi^*(0, 1.8) = 0.0695$ m (the same water level but above $y = 2.4$ m), values agreeing with 2.4695 and 0.0695 for the triplet at (0, 1.8). Each triplet's numbers are related (see text) by $\phi + V - 2^*_\phi = 1.9888$, that is, by bottom + top − 2 (middle) = 1.9888; thus, for point (0, 1.8) one has $2.4695 + (-0.3417) - 2(0.0695) = 1.9888$, and for point (3, 0) one has $3.0943 + 0.2831 - 2(0.6943) = 1.9888$.

obtained are given in Table 1, which requires explanation. First we recognize, by (54), that z is directly proportional to e/k for any point (x, h). Choosing $e/k = 0.2$ as in Figure 4 and $e = 0.01$ we determine k to be 0.05. With $e = 0.01$, we can get a direct proportion multiplier $(1/k)/(1/0.05) = 0.05/k$ (second column from the left in Table 1) to multiply into the known z values for $k = 0.05$, and thus get z values for the other values of k. For example, to get z for $x = 0.6$ and $k = 0.025$ m/d in the table, we see that the direct

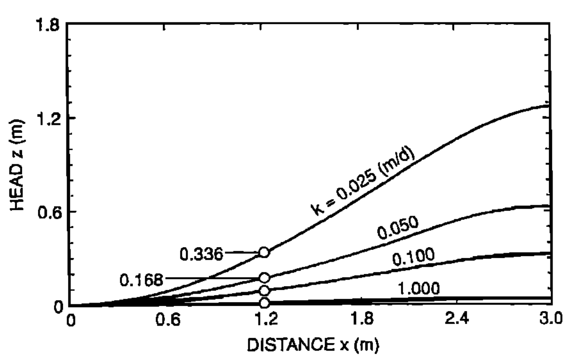

Fig. 5. Water table height z versus x for several values of k when $e = 0.01$ m/d, $f = 0.4$, $s = 3$ m, $h = 2.4$, $\beta = 1.0$, $b = 1.4$, $\rho = 0.05$, and $r = 0.0375$ m; example values of z at $x = 1.2$ m are $z = 0.0084$, 0.0840, 0.1680, and 0.3360 m.

proportion multiplier $0.05/k$ becomes $0.05/0.025 = 2$, so that z for $k = 0.025$ at $x = 0.6$ becomes $2(0.0463) = 0.0926$, and similarly for other values of z in Table 1.

The z values of Table 1 as displayed in Figure 5 bring out a main conclusion of the theory: For $e, f, s, h, \beta, b, \rho$, and r all constant, doubling k halves z; increasing k tenfold decreases z to one-tenth; increasing k to "infinity" as by changing clay to gravel, makes z "zero"; all this, while maintaining e constant, as $e = [[1 - f]/s]Q/2$ by (1) and (2), and hence $Q/2 = es/(1 - f)$ constant. The last sentence is true for H as well as for z, because H is a special case of z.

We come next (see below (51b)) to the curves in Figure 6 of h^*_w, h^*_0, and H all versus s. For these curves we have used $f = 0.4$ and $f = 0$ (for $f = 1$ see subsection 5.8.3), s as $s = 2, 2.5, 3, 5, 10, 20$, and 40 m, $h = 2.4$, $\beta = 1.0$, $b = 1.4$ m, and (for h^*_w and h^*_0) $\rho = 0.05$ and $r = 0.0375$ m. The head H does not need radii values (see (56a)–(56f)) except that the radii should be small as noted in the footnote to Table 1.

A useful way to present the asterisked heads h^*_0 and h^*_w and H is to show them normalized with respect to e/k. Figure 6 "shows at a glance" how spacing s influences the normalized heads. But for better values especially of h^*_w and for further comparisons see the microfiche supplement. Physically, the head h^*_w may be negative. When it is, the levels in Figure 2 of the ditch bottom, the drain pipe end, and the ditch water will all be below the level $y = h$.

TABLE 1. Points for z Versus x of Figure 5

k, m/d	$0.05/k$	z, m					
		$x = 0$ m	$x = 0.6$ m	$x = 1.2$ m	$x = 1.8$ m	$x = 2.4$ m	$x = 3.0$ m
0.025	2	0.0	0.0926	0.3360	0.6802	1.0578	1.2572
0.05	1	0.0	0.0463	0.1680	0.3401	0.5289	0.6286
0.1	0.50	0.0	0.0232	0.0840	0.1701	0.2645	0.3143
1.0	0.05	0.0	0.0023	0.0084	0.0170	0.0264	0.0314

Here, $\phi^*(x, h) = z$ of (59*b*), and parameters are $e = 0.01$ (m/d), $f = 0.4$, $s = 3.0$ (m), $h = 2.4$ (m), $\beta = 1.0$ (m), $b = 1.4$ (m), and ρ and r are any values that are sufficiently small (with their corresponding heads) to approximate radii of circles.

5.8. *Further Unknowns*

5.8.1. *Drain size and circularity.* Figure 3 shows that equipotentials of approximately circular shape surround the tube axis, $(0, \beta) = (0, 1.0)$ (at the left) and at $(s, b) = (3, 1.4)$ (at the right)). The smaller the circles, the closer they become to exact circles with centers at the tube axes. To see how closely the smallest "circles," the drain tube "circles" in Figure 3, (of radius $\rho = 0.05$ m at the left, and of radius $r = 0.0375$ m at the right) approximate true circles, we have for convenience calculated for the left tube, potentials above, to the right of, and below the tube axes, by use of (53*a*) and (29*a*)–(29*h*). We found for the left tube (for the parameters (59*a*) of Figure 3)

$$\phi(0, \beta + \rho) = \phi(0, 1.05) = 2.2070$$

$$\phi(\rho, \beta) = \phi(0.05, 1.00) = 2.2100$$

$$\phi(0, \beta - \rho) = \phi(0, 0.95) = 2.2120$$

Similarly, we found for the right tube

$$\phi(s, b + r) = \phi(3.0, 1.4375) = 4.0579$$

$$\phi(s - r, b) = \phi(2.9625, 1.40) = 4.0599$$

$$\phi(s, b - r) = \phi(3.0, 1.3625) = 4.0622$$

The numbers in groups are, for practical purposes, equal rounded at the second decimal place. Each value in the first group is $\phi = 2.21$ m and each value in the second group is $\phi = 4.06$ m.

5.8.2. *Stagnation point, sensitivity tests.* The height, say y_S, of the stagnation points S (Figure 3) may be calculated from

$$\partial\phi(0, y_S)/\partial y = 0$$

We found $y_S = 1.73$ m in Figure 3. To see how, say, H varies with s when h, β, and b are constant, we may obtain $\partial H/\partial s$ from (56*a*), analytically, or approximately, by the quotient $(H_2 - H_1)/(s_2 - s_1)$, by digital computer, for $(H_2 - H_1)$ and $(s_2 - s_1)$ sufficiently small (here sufficiently small ordinarily means that a larger denominator in the quotient with its corresponding new value of numerator gives the same result to the desired number of significant figures as the previous quotient gives).

5.8.3. *Formulas for* $f = 1$, $f = 0$, *and* $f < 0$. When f is 1 we have e equal to zero and the flow $Q/2$ in Figure 2 all goes to the drain tube. Then, by (1), (2), and (26*b*), we replace in our equation for ϕ, h_0, h_w, z, H, ϕ^*, h_0^*, and h_w^* the factor

$$(e/k)s/(1 - f) = \psi_0/k \qquad (60)$$

by the factor $(Q/2)/k$, and replace f (where f appears otherwise) by unity. For example, to get H of (56*a*), for $f = 1$, we replace $(e/k)s/(1 - f)$ by $(Q/2)/k$ and in (50) and for A, B, C, D, and E in (56*b*)–(56*f*) set $f = 1$. Note that when we have $f = 1$ we must consider $Q/2$ and k as knowns for the flow region and that the parameter e does not occur in the analysis.

For $f = 0$ there are no difficulties with our formulas. They all work. The parameter β then vanishes from the analysis and there is no sink at $(0, \beta)$, as is seen by inserting $f = 0$ in (29*e*) and (29*h*). The most interesting aspect, perhaps, of the condition $f = 0$ is that it describes subirrigation by tubes $2s$ apart and at height b above the barrier; all water goes to evapotranspiration, none to drainage. The theory can be modified to work for negative values of f. Then, both tubes supply subirrigation water and there is no drainage.

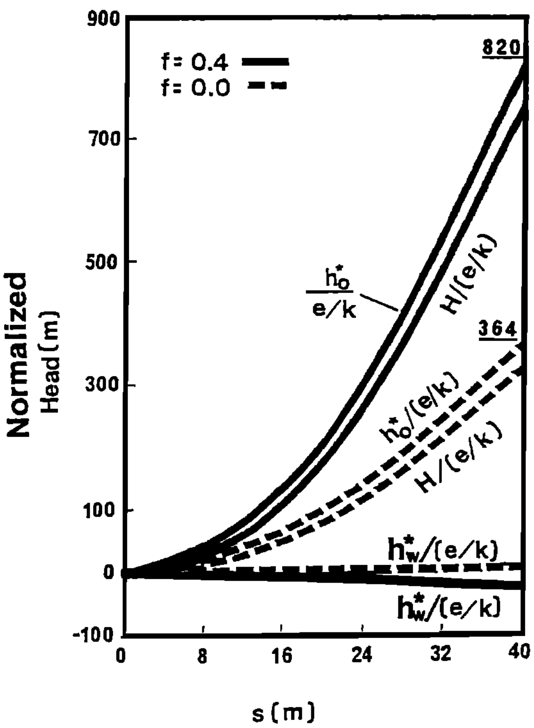

Fig. 6. Normalized heads $h_w^*/(e/k)$, $H/(e/k)$, and $h_0^*/(e/k)$ versus s for $f = 0.4$ and $f = 0$ and for $h = 2.4$, $\beta = 1$, $b = 1.4$, $\rho = 0.05$, and $r = 0.0375$ m. Multiply ordinates by the numerical value of e/k to get the heads in meters. For example, to get the real head from the normalized head $h_0^*/(e/k) = 820$ m for $f = 0.4$, $s = 40$ m, $e = 0.01$ m/d and $k = 100$ m/d (coarse sand), get $h_0^* = 820(0.01/100) = 0.082$ m; and for $f = 0$, and for the same $s = 40$ and $e/k = (0.01)/100$, get the corresponding real value $h_0^* = 364(0.01/100) = 0.036$ m.

TABLE 2. Variation of Heads With s

s, m	$h_w^*/(e/k)$, m	$H/(e/k)$, m	$h_0^*/(e/k)$, m
0	0.0	0.0	0.0
2	−0.3630	1.1040	4.7773
3	−0.9505	3.1435	8.2995
5	−2.1858	10.1267	18.2092
10	−5.3550	44.5653	59.9383
20	−11.7000	186.3467	216.3067
40	−24.4067	761.5733	820.7067

The heads in this table are for the solid curves $f = 0.4$ of Figure 6. Values for the parameters are $f = 0.4$, $h = 2.4$, $s = 0$–40, $\beta = 1.0$, $b = 1.4$, $\rho_\cdot = 0.05$ and $r = 0.0375$ m. The asterisk on h_w^* and h_0^* means the (normalized) heads are measured positively upward with respect to the level $y = h$ of Figure 2; H is also so measured but is not marked with an asterisk. For the shown heads a volume of water $Q/2$ flows from the irrigation tube into the soil for a unit 1-m length of irrigation pipe in a day and is determined from the equation $Q/2 = es/(1 - f)$; the corresponding volume of water that goes to the drain tube is determined from the equation $fQ/2 = fes/(1 - f)$.

5.8.4. *Streamline preparation.* Figure 3 has streamlines and equipotentials. Streamlines are needed for transport times as well as for other features. Streamlines of Figure 3 are determined by (5), (6), (19), and (20). When summations were slowly converging, we speeded up the convergence, as for the potential function, by use of appropriate formulas. Two such, derivable from *Mangulis* [1965, p. 97, equation (2)], *Kirkham* [1958, equation (96)], or *Jolly* [1961, equation (594)], are obtained as (with $m = 1, 2, \cdots$),

$$\sum (1/m) \sin m\theta e^{-mE} = \tan^{-1}[(\sin\theta)/(e^E - \cos\theta)] \qquad (61)$$

$$\sum (1/m)(-1)(\cos m\pi) \sin m\theta\, e^{-mE} = \tan^{-1}[(\sin\theta)/(e^E + \cos\theta)] \qquad (62)$$

Formula (61) is derived by *Kirkham* [1958]; formula (62) is obtained by inserting $\theta = (\pi - \theta)$ in (61) and simplifying. Formulas (61) and (62) are valid for all θ and for $0 < E < \infty$.

We note also that streamlines are equally spaced along the top of the flow region and are like the spokes of a wheel at the tube source and sink.

5.8.5. *Seepage velocities, times of solute transport.* The Darcy seepage velocity v at a point (x, y) in Figure 2 may be expressed (with $d\phi$ being an increment of head for a distance ds along a streamline)

$$v = -k\, d\phi/ds = [(-k\partial\phi/\partial x)^2 + (-k\partial\phi/\partial y)^2)]^{1/2} \qquad (63)$$

where $\partial\phi/\partial y$ and $\partial\phi/\partial y$ can be obtained from (26c) and its auxiliary equations (29a) and (38)–(45). In doing the partial differentiation of (63) the constant $V(0, h)$ in (26c) drops out. By use of (63), one obtains, similar to *Kirkham and Affleck* [1977], the time of travel for solute that moves with the water from one point on a streamline to another.

5.8.6. *A table of normalized heads versus spacing s.* Table 2 gives better values of heads to four decimal places than can be read from the solid curves $f = 0.4$ of Figure 6 for computer program comparisons.

As an example of Table 2 we may compare the maximum water table elevation (arch height), H, for spacings of, say, 10 and 40 m when we have $e = 0.01$ m/d and $k = 10$ m/d to give $e/k = 0.001$.

From the table and our value of e/k we find that for $s = 10$ we have $H = 44.5653(0.001) = 0.0446$ m and for $s = 40$ we have $H = 761.5733(0.001) = 0.7616$ m, which is an increase of $0.7615 - 0.0445 = 0.7170$.

If we wish to know the increase for a spacing change from, say, 16 m to 40 m, we may make a graph of $H/(e/k)$ versus s from the first and third columns from the left of Table 2 for $f = 0.4$ and read from the graph for $s = 16$ m the value $H/(e/k) = 115$ m. So, the change in elevation of H for going from $s = 16$ m to $s = 40$ m is $(761.5733 - 115)(0.001) = 0.641$, or approximately 0.64 m for $e/k = 0.01/10$ and the parameters of Table 2. (The number 0.64 is given in the abstract.)

5.8.7. *Correction for "gravel" between water table "strips".* At the end of subsection 2.2 a "correction" factor for heads z and H is mentioned. This factor (which is to be multiplied into the right-hand side of an equation for z or H, as for H of (56a)], is [*Kirkham and Powers*, 1984, p. 120]

$$[1 - (e/k)]^{-1} \qquad (64)$$

Since the ratio e/k will generally be 0.01 or smaller, the correction generally will not be more than $(1 - 0.01)^{-1} = 1.01$ approximately and can be neglected. The factor (64) only partly corrects: It gives a mathematically exact upper limit for z and H.

5.8.8. *Heads and Q/2 for s/h* $\gg$ 1. For brevity we shall consider only H for large s. The procedure for other heads such as ϕ, h_e, and h_w will be similar. We shall work from (56a) with its auxiliary quantities $A, B, \cdots, E$.

For any large enough L we have

$$\sinh L = \cosh L = (1/2)e^L \qquad \coth L = (\tanh L)^{-1} = 1$$

so that $A, B, \cdots, E$ in (56a) become, when we neglect cosine values (-1 to $+1$) compared to e^L, etc., the approximate expressions

$$A = (2/\pi)(1 + f) \ln [(s/h)/(e^{\pi s/2h}/2)]$$

$$B = (1 + f)(s/h) - (1/\pi) \ln (2^{2(f+1)} \cos^{2f} \pi\beta/2h \cos^2 \pi b/2h)$$

See Appendix C to get the constant B approximation. There seems to be no simple expression for C for large s/h; $D = 0$; $E = 0$.

5.8.9. *Heads and Q/2 for large barrier depth h.* Large barrier depth h compared to s does not seem to have field application. For large h/s, Fourier integral type solutions will occur for heads. A Schwartz-Christoffel transformation solution would need four angle jumps, two of $\pi/2$ and two of π each with ψ as the potential. If both tubes are at the same elevation and of equal strength, the work will be simplified.

5.8.10. *Management of f.* When e, k, s, h, β, b, ρ, and r are given, then h_0 and h_w can be calculated for a desired f. If the f does not have the desired value, a different f can be obtained by changing, say, s. We do not know a simple procedure for selecting parameters from those that can be changed except trial and error using additional figures like Figure 6 (see the microfiche supplement). Iteration or a method described by *Walczak et al.* [1988] might be used to solve for f when everything else in an equation containing f is given. The height h in Figure 2 might be taken as an unknown.

6. AN EXAMPLE CALCULATION OF H

Remembering that radii ρ and r are not needed to calculate H (see discussion below (26c)), we take values of parameters from (59a) (conveniently) as

$$e/k = 0.2 \qquad f = 0.4 \qquad s = 3 \qquad h = 2.4 \qquad \beta = 1.0 \qquad b = 1.4 \tag{65}$$

Equations (65) and (56*a*)–(56*f*) then give

$$H = [(e/k)s/(1-f)][V(s, h) - V(0, h)] \tag{66}$$

$$H = [(e/k)s/(1-f)][A + B + C + D + E] \tag{67}$$

$$H = [(0.20)(3)/(1-0.4)][A + B + C + D + E] \tag{68}$$

$$H = (1.0)(A + B + C + D + E) \tag{69}$$

$$A = (2/\pi)(1 + 0.4) \ln \{[(3/2.4)/\sinh 3\pi/4.8]\}$$

$$A = -0.9156$$

$$B = (0.4/\pi) \ln[(\cosh 3\pi/2.4 + \cos 1.0\pi/2.4)/(2 \cos^2 1.0\pi/4.8)] + (1/\pi) \ln[(\cosh 3\pi/2.4 + \cos 1.4\pi/2.4)/(2 \cos^2 1.4\pi/4.8)]$$

$$B = 0.3838 + 1.1215 = 1.5053$$

$$C = (4/\pi)(1/1)(1.4)\{[\coth 1\pi(2.4)/3] - 1\} + (4/\pi)(1/3)(1.4)\{[\coth 3\pi(2.4)/3] - 1\} + \cdots$$

$$C = 0.0235 + 0.0000004 + \cdots = 0.0235$$

$$D = \frac{4}{\pi}\frac{1}{1}\left[\cos^2 \frac{1\pi(1.4)}{4.8} + 0.4 \cos^2 \frac{1\pi(1.0)}{4.8}\right] \cdot \exp\left(-\frac{1\pi(3)}{2.4}\right) \tanh \frac{1\pi(3)}{4.8} + \frac{4}{\pi}\frac{1}{3} \cdot \left[\cos^2 \frac{3\pi(1.4)}{4.8} + 0.4 \cos^2 \frac{3\pi(1.0)}{4.8}\right] \cdot \exp\left(-\frac{3\pi(3)}{2.4}\right) \tanh \frac{3\pi(3)}{4.8} + \cdots$$

$$D = 0.0150 + 0.00000296 + \cdots = 0.0150$$

$$E = \frac{4}{\pi}\frac{1}{2}\left[\sin^2 \frac{2\pi(1.4)}{4.8} + 0.4 \sin^2 \frac{2\pi(1.0)}{4.8}\right] \cdot \exp\left(-\frac{2\pi(3)}{2.4}\right) \tanh \frac{2\pi 3}{4.8} + \cdots$$

$$E = 0.0003$$

So, from the values of A, B, C, D, and E we find

$$A + B + C + D + E = -0.9156 + 1.5053 + 0.0235 + 0.0150 + 0.0003 = 0.6285 \tag{70a}$$

a value that we can check against Figure 4, as follows.

Comparing (66) and (67) we have (see also (50))

$$A + B + C + D + E = V(s, h) - V(0, h) \tag{70b}$$

But in Figure 4, which is for the same parameters as in (65), we find in the upper right-hand corner

$$V(s, h) = 0.2174$$

and in the upper left-hand corner

$$V(0, h) = -0.4112$$

so that Figure 4 gives

$$V(s, h) - V(0, h) = 0.2174 - (-0.4112) = 0.6286 \tag{71}$$

So, our calculated result (70), namely, 0.6285, agrees with our Figure 4 value, namely 0.6286, to within rounding, a check.

We may make two more checks. For the first we put the value of the (dimensionless) sum

$$A + B + C + D + E = 0.6285$$

of (70) in (69) and find

$$H = (1.0)(0.6285) \text{ m} \tag{72}$$

which is a check on the value of H as stated in the caption of Figure 3. For the remaining check we look at Figure 5 and its caption and see then that the curve 0.050 for $k = 0.050$ m/d is just the curve for parameters (65) used to verify (72). We now read from Figure 5 for the 0.050 curve at $s = 3$ ($s = 3$ is in (65)) the value $H = 0.63$ m, which checks our value of $H = 0.6286$ (a value also found in the entries in Figure 4), our last check.

7. SUMMARY AND CONCLUSIONS

Tube subirrigation with drainage can be and has been analyzed by mathematics and physics to show and predict how various parameters govern the soil water flow. Principal parameters are tube spacing and depth, soil depth and hydraulic conductivity, and the evapotranspiration coefficient. Working formulas are derived that give hydraulic head and seepage velocities for any and all points of the flow medium, including its boundaries.

APPENDIX A

The normalized flow net (Figure 3) is valid for $e/k = 0.20$ but is also valid for other values of e/k (see the paragraph below (59*a*)).

A proof is needed "for other values."

We let ϕ_n and ψ_n denote normalized potential and normalized stream functions. Then, with subscripts min and max referring to minimum and maximum we have by definition ϕ_n as

$$\phi_n = [\phi - \phi_{\min}]/[\phi_{\max} - \phi_{\min}] \tag{A1}$$

and ψ_n as

$$\psi_n = [\psi - \psi_{\min}]/[\psi_{\max} - \psi_{\min}] \tag{A2}$$

where the right-hand sides need to be put in terms of the notation of the problem at hand (Figure 2).

We first consider (A1).

In Figure 2 at the left we have $\phi_{\min}$ given by

$$\phi_{\min} = h_w \tag{A3}$$

with h_w being the head in the drain outflow tube; and in Figure 2, at the right, we have $\phi_{\max}$ given by

$$\phi_{\max} = h_0 \tag{A4}$$

with h_0 being the head in the irrigation tube. Therefore, from (A1), (A3), and (A4), we have

$$\phi_n = \frac{\phi - h_w}{h_0 - h_w} \tag{A5}$$

where the terms ϕ, h_w and h_0, which are practical, need here to be put in function notation.

Equation (26c) gives the head ϕ at any point (x, y) in function notation, as

$$\phi = h + (e/k)[s/(1-f)][V(x, y) - V(0, h)] \tag{A6}$$

Similarly, (26g) gives h_w of (A5) as

$$h_w = h + (e/k)[s/(1-f)][V(s-r, \beta) - V(0, h] \tag{A7}$$

and (26h) gives h_0 as

$$h_0 = h + (e/k)[s/(1-f)][V(s-r, b) - V(0, h)] \tag{A8}$$

We put values of (A6), (A7), and (A8) in (A5) and simplify by canceling first four h terms, the four $[(e/k)[s/(1-f)]$ terms, and the four $[V(0, h)]$ terms to find ϕ_n as

$$\phi_n = [V(x, y) - V(s-r, \beta)]/[V(\rho, \beta) - V(s-r, b)] \tag{A9}$$

which does not depend on (e/k), because, by (23) we see that $V(x, y)$ (in all its forms) does not depend on e/k.

There is a further conclusion. We look at (23) where in the first summation of its right-hand side we see a cos cosh/sinh combination. We now think of the cos term, cos $(m\pi x/s)$ as

$$\cos(m\pi x/s) \rightarrow \cos(m\pi x/h)/[1/(s/h)] \tag{A10}$$

and, similarly from the cosh and sinh terms. Then, we see that (23) may be written as

$$V(x, y) = F(x/h, y/h, f, s/h, h/h, \beta/h, b/h) \tag{A11}$$

where all terms in the right-hand side are dimensionless and hence independent of the unit used in getting $V(x, y)$. The conclusion is that the equipotential lines in Figure 3 are valid for any unit of length used in getting them. For $V(x, y)$ as shown in (A11), taking the flow domain height, h, as 1 m would be convenient.

We come now to (A2). We wish to show that the normalized streamlines do not depend on e/k.

For (A2) we need values of ψ, $\psi_{\min}$ and $\psi_{\max}$. In Figure 2 we see that $\psi_{\min}$ is given by

$$\psi_{\min} = 0.0\psi_0 \tag{A12}$$

and $\psi_{\max}$ by

$$\psi_{\max} = 1.0\psi_0 \tag{A13}$$

where ψ_0 is, by (1), given as

$$\psi_0 = Q/2$$

and alternatively ψ_0 is, by (2) given as

$$\psi_0 = (e/k)[s/(1-f)] \tag{A14}$$

We put the right-hand sides of (A12) and (A13) in (A2) and thus find ψ_n as

$$\psi_n = \psi/\psi_0 \tag{A15}$$

Now, for (A15), we have the right-hand side given by (5) where the constants a_m, b_m and c_m are given by (6)–(8) for slit tubes and by (6), (19) and (20) for circle tubes. In the right-hand side of (5), and of (6)–(8), (19) and (20) the ratio e/k does not occur. Therefore the normalized stream function ψ_n of (A15) and (A2) does not depend on (is invariant with) e/k.

Our remarks about length units for ψ_n apply also to ϕ_n. The remarks apply to an assumed homogeneous isotropic flow medium.

APPENDIX B: A CHECK RELATION FOR FIGURE 4

The relation to be derived is (see text following (59b)):

$$V + \phi - 2\phi^* = 1.9888 \tag{B1}$$

We are given (see top of Figure 4)

$$\psi_0/k = 1 \tag{B2}$$

$$h = 2.4 \tag{B3}$$

$$V(0, h) = -0.4112 \tag{B4}$$

We proceed to find (B1) as

$$V = V(x, y) \tag{B5}$$

(an identity),

$$\phi = h + V - V(0, h) \tag{B6}$$

from (25d) and (B1),

$$V - V_0 = \phi - h \tag{B6$'$}$$

from (86),

$$\phi^* = \phi - h \tag{B7}$$

from (26i) (or (53b)), and

$$\phi - \phi^* = h \tag{B7$'$}$$

from (B7). We multiply (B7) by (−2) as

$$-2\phi^* = -2(\phi - h) \tag{B8}$$

We add the left- and right-hand sides of (B5), (B6) and (B8) and equate the sums and manipulate (using B6′))

$$\begin{aligned} V + \phi - 2\phi^* &= V + [h + V - V(0, h)] - 2(\phi - h) \\ &= 2[V - V(0, h)] + V(0, h) + h - 2(\phi - h) \\ &= 2(\phi + h) + V(0, h) + h - 2(s - h) \\ &= V(0, h) + h \end{aligned} \tag{B9}$$

By (B4) and (B3) this gives

$$V + \phi - 2\phi^* = -0.4112 + 2.4 = 1.9888 \qquad (B10)$$

That is, we have with -0.4112 from (38)

$$V + \phi - 2\phi^* = 1.9888$$

which is (B1).

Note that this appendix proves a relation useful in proofreading Figure 4 by a reader to assure him that he can use Figure 4 with impunity to check a digital computer program. Figure 4 has served the authors to keep them on track in equation derivations.

APPENDIX C: THE CONSTANT B OF SECTION 5.8.8

We wish to prove that the constant B of section 5.8.8 can be derived from (56c), when we impose the condition that s/h or L of section 5.8.8 is large.

Proof: Equation (56c) is

$$B = \frac{f}{\pi} \ln \frac{\cosh \frac{\pi s}{h} + \cos \frac{\pi \beta}{h}}{2 \cos^2 (\pi\beta/2h)} + \frac{1}{\pi} \ln \frac{\cosh \frac{\pi s}{h} + \cos \frac{\pi b}{h}}{2 \cos^2 (\pi b/2h)} \qquad (C1)$$

For large s/h, (C1) becomes approximately

$$B = \frac{f}{\pi} \ln \frac{\frac{1}{2} e^{\pi s/h}}{2 \cos^2 (\pi\beta/2h)} + \frac{1}{\pi} \ln \frac{\frac{1}{2} e^{\pi s/h}}{2 \cos^2 (\pi b/2h)}$$

which with careful manipulation gives alternatively

$$B = (1 + f) \frac{s}{h} - \frac{1}{\pi} \ln \left[2^{2(f+1)} \left(\cos^{2f} \frac{\pi\beta}{2h} \right) \left(\cos^2 \frac{\pi b}{2h} \right) \right] \qquad s/h \to \infty \qquad (C2)$$

As an example of (C2) versus (C1), we take parameters as $s = 10$, $h = 2.4$, $f = 0.4$, $\beta = 1$, and $b = 1.4$. Equation (C1) then gives for the exact equation $B = 5.590473768$, and (C2) gives for the approximate equation $B = 5.590473975$, which is the same as for (C1) to six decimal places.

Acknowledgments. We thank Jamie Barmettler for computing and Phyllis Murphy for typing. Some aspects of this work have been presented at an International Symposium, Groundwater Management Quantity and Quality, sponsored by the International Association of Scientific Hydrology, Benidorm (Valencia) Spain, October 1989 (D. Andreu, Secretary of the Symposium, Universidad Politecnica, Valencia, Spain). Journal paper J-13907 of the Iowa Agriculture and Home Economics Experiment Station, Ames, Iowa; projects 2556 and 2715.

REFERENCES

Brandyk, T., and T. G. Wesseling, Soil moisture flow in drainage-subirrigation system, *J. Irrig. Drain. Eng.*, *113*(1), 86–97, 1987.
Cooper, J. R., and J. L. Fouss, Rainfall probability forecasts used to manage a subdrainage-subirrigation system for watertable control, *Agric. Water Manage.*, *15*, 47–59, 1988.
Criddle, W. D., and C. Kalisvaart, Subirrigation systems, *Agronomy*, *11*, 905–921, 1967.
Dwight, H. B., *Tables of Integrals and Other Data*, 4th ed., Macmillan, New York, 1961.
Ernst, L. F., Grondwaterstromingen in de verzadigde zone en hun berekening bij de aanwezigheid van horizontale evenwijdige open leidengen, *Versl. Landbouwkd. Onderz.*, *67*(15), 189 pp., 1962.
Jolly, L. B. W., *Summation of Series*, Dover, New York, 1961.
Kaplan, W., *Advanced Calculus*, pp. 315, 326, Addison-Wesley, Reading, Mass., 1959.
Kirkham, D., Seepage of steady rainfall through soil into drains, *Eos Trans. AGU*, *39*(5), 892–908, 1958.
Kirkham, D., and S. B. Affleck, Solute travel times to wells, *Ground Water*, *15*, 231–242, 1977.
Kirkham, D., and R. Horton, Subirrigation with drainage for drought prone areas, in *Proceedings, International Conference on Water Resources and Needs and Planning in Drought Prone Areas*, part 2, edited by I. M. Abdel-Magid, M. A. Khadam, M. A. Osman, and G. M. Abdo, pp. 957–980, UNESCO, Paris, 1986.
Kirkham, D., and W. L. Powers, *Advanced Soil Physics*, revised ed., Robert E. Krieger, Malabar, Fla., 1984.
Mangulis, V., *Handbook of Series for Scientists and Engineers*, Academic, San Diego, Calif., 1965.
Melvin, S. W., R. S. Kanwar, and D. G. Baker, Evaluation of a dual-level subirrigation system, paper presented at the 3rd National Irrigation Symposium, Am. Soc. of Agric. Eng., Phoenix, Ariz., 1990.
Skaggs, R. W., Water movement factors important to the design and operation of subirrigation systems, *Trans. ASAE*, *24*, 1553–1561, 1981.
Thornton, K. O., Irrigation system, *U.S. Patent 4,538,377*, U.S. Dep. of Commerce Patent and Trademark Off., Washington, D. C., Sept. 3, 1985.
Van Bakel, P. J. T., Using drainage systems for supplementary irrigation, *Irrig. Drain. Syst.*, *2*, 125–137, 1988.
Walczak, R. T., R. R. Van der Ploeg, and D. Kirkham, An algorithm for the calculation of drain spacing for layered soils, *Soil Sci. Soc. Am. J.*, *52*(2), 336–340, 1988.
Willardson, L. S., Methods of irrigation, Reports Prepared for a Conference on Agricultural and Water Resources Development in the Arab World, pp. 34–38, Iowa State Univ., Ames, 1987.

R. Horton and D. Kirkham, Department of Agronomy, Iowa State University, Ames, IA 50011.

(Received February 26, 1990;
revised September 19, 1991;
accepted September 25, 1991.)